especially suitable. The non-specialist who wishes an introduction to mathematics can profit less by trigonometry or algebra than by an immediate insight into the theory of functions. Unity is to him of more importance than any other element. The second type is the engineering student who during his freshman year needs an introduction to calculus and to vectors, as a preparation for physics and mechanics, rather than most of the work in college algebra. Partial fractions are best considered preliminary to certain integrations; the binomial theorem fits into the study of series; permutations and combinations are only of the most indirect value. The third type is the student who would later specialize in mathematics, and for him the present outline includes everything that he now gets in his freshman courses of trigonometry, algebra and analytics with the exception of permutations, a loss more than counterbalanced by his knowledge of calculus.

THE DUPLICATION PROBLEM.[1]

By JAMES H. WEAVER, West Chester High School.

There have come down to us from the remote past three problems of perennial interest, namely, the duplication problem, the trisection problem, and the quadrature problem. The first of these has for its object the finding of the edge of a cube that is double a given cube, the second the trisection of any angle, and the third the finding of a square equivalent to a given circle. It is the object of this paper to give:

I. A short historical sketch of the duplication problem, calling attention to the various methods of attacking the problem that were made possible by the advancement of mathematics.

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[1] The following authorities may be consulted for a fuller discussion of the problem:

(*a*) *Collections of Pappus*, ed. Hultsch, Berlin, 1876, page 31 and ff. In addition to the approximate solution mentioned in the text, this contains the principal solutions of the Greeks of the Alexandrian School.

(*b*) *Historia Problematis de Cubi Duplicatione*, by N. T. Reimer, Göttingen, 1798, 8vo, 238 pages. A very full account of the history of the problem.

(*c*) *Historia Problematis Cubi Duplicandi*, by C. H. Biering, Copenhagen, 1844, 4to, 64 pages. This was largely stolen from (*b*) according to S. Günther. See Cantor, *Geschichte der Mathematik*, Volume 4, pages 28–9.

(*d*) W. W. R. Ball, *Mathematical Recreations and Essays*, New York, Sixth Edition, 1914, pages 285–291. The discussion here is merely an outline of a few of the most noted solutions with references to the original sources.

(*e*) *Das Delische Problem*, A. Sturm, Linz, 1895–7. 4to, 140 pages. A full discussion of the problem in all its stages of development.

(*f*) Cantor, *Geschichte der Mathematik*, Vol. I, first edition, page 349. We have here an authoritative historical account of some of the solutions of the Greeks.

(*g*) Article by A. Conti, found in the following books: *Questioni Reguardanti le Matematiche Elementari*, Volume 2, Bologna, 1914, pages 185–231, and *Fragen der Elementar Geometrie*, Theil 2, Leipzig, 1907. Both these books were edited by Enriques. This article gives a very good discussion of some approximate solutions and takes up the question of the impossibility of a solution by means of ruler and compasses.

(*h*) *Famous Problems of Elementary Geometry*, F. Klein, translated by Beman and Smith, Boston, 1897. This volume includes a discussion of the impossibility of the problem by means of ruler and compasses. We will refer to the above mentioned accounts by the names of the authors; thus, Sturm, p. . . ., etc.

II. Solutions by means of the conchoid and the cissoid.

III. An approximate solution, recorded by Pappus, illustrating the difficulty under which the Greeks labored when dealing with problems of this type.

I. Historical Sketch.

Just how the duplication problem originated is not known. It probably dates back to the early Pythagoreans (about 530 B.C.) who had succeeded in finding the side of a square that was double a given square, and proving that the sides of the two squares are incommensurable. After this accomplishment it would be the natural thing to attempt to find the edge of a cube that is double a given cube. But tradition has given the origin of the problem a romantic setting. Eutocius (about 480 B.C.)[1] has preserved for us one version of the story which was related by Eratosthenes (276–194 B.C.) in a letter to king Ptolemy. It is as follows:

"It is related that one of the old tragic poets whom Minos had imported says that when Minos wished to erect for his son Glaucus a tomb and noticed that its dimensions were one hundred feet (ἑκατομ πεδος) on all sides he exclaimed: 'You have enclosed too small a space for a royal tomb. Double it, but forget not the beautiful form. Therefore double each edge of the monument.'

"But he had clearly erred. For by doubling the edges, the surface is made four times as great and the volume is increased eight fold. Nevertheless it made the question of how a body could be doubled without changing its form an object of investigation among geometers, and this is called the duplication of the cube. That is, they set up a given cube and sought to double it."

The first real progress in the solution of the problem was made by Hippocrates of Chios (about 420 B.C.). He reduced it to the one of finding two mean proportionals to two given lines, a form in which it has since been stated.[2] He did not however succeed in finding the mean proportionals. This task was left to Archytus of Tarentum who (about 400 B.C.) accomplished it by means of the intersections of solids.[3] Soon after this Menaechmus (about 340 B.C.), probably

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[1] Archimedes, *Opera Omnia cum Commentariis Eutocii*, ed. Heiberg, Leipzig, 1880–1, volume 3, page 102 ff.

[2] Hippocrates noted that if two lines a and $2a$ were given and if two mean proportionals could be inserted between them such that

$$a : x = x : y = y : 2a,$$

then $x^2 = ay$ and $y^2 = 2ax$, from which it readily appears that $x^3 = 2a^3$.

[3] The following is an outline of the solution of Archytus. Let AD be the larger of the two given lines. Then on AD as diameter describe a circle and on this circle as base erect a cylinder. From A in the circle draw the chord AB equal to the smaller of the two given lines and extend this chord until it meets the tangent drawn from the point D. Let this point be P. Then let the triangle APD be revolved about AD as axis generating a cone. Then suppose a semicircle drawn on AD as diameter and perpendicular to the circle ADB, and let this semicircle be so moved that A remains fixed and the semicircle remains perpendicular to the circle ABD. It will then generate a solid. The intersection of the three solids just described will determine a point. From this point draw an element of the cylinder, and let this cut the circle ABD in C. Draw AC. It can then be easily proved that $\overline{AC}^3 = 2\overline{AB}^3$.

following the suggestion of Archytus,[1] discovered the conic sections and used them to give two solutions of the duplication problem.[2] Plato (about 340 B.C.) contrary to his usual custom of dealing with such problems produced a mechanical solution.[3] Nicomedes invented for this purpose the conchoid[4] and Diocles at about the same time (second century B.C.) produced the cissoid.[5] All the other solutions given by the Greeks were either mechanical or depended on the conic sections.[6]

After the decline of Greek geometry nothing was done to advance the problem until the sixteenth century A.D. when Stifel (1486–1567) attacked it from the side of number theory.[7] Then Viète in his geometry (1593) pointed out the fact that every cubic or biquadratic that is not otherwise reducible leads to either the duplication or the trisection problem when solved.[8] Shortly after this Claud Richard records (1645) some solutions that he attributes to Christopher Grienberger, a contemporary, and which employ a new curve called the proportionatrix.[9] Other solutions of a geometric character were given by Grégoire de Saint-Vincent (1647), Newton (1642–1727) and Huygens (1625–1695).[10]

But the answer to the question of the impossibility of a solution of the problem by means of straight lines and circles was finally given by Descartes (1596–1650) in his geometry.[11] In Book II of this work Descartes classified problems as did the Greeks, into plane, solid and linear[12] and proved that, algebraically considered, plane problems correspond to equations of the first degree and of the second degree, solid problems to equations of the third degree, and linear problems to equations of the fourth and higher degrees.

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[1] In his solution Archytus made use of a point determined by the intersections of solids, one of which was a cone. The locus of such a point could be considered as a curve on a cone. This would naturally lead to the investigation of the properties of curves on a cone.

[2] The two solutions of Menaechmus consist analytically in finding (1) the intersection of the two parabolas $x^2 = ay$ and $y^2 = 2ax$ other than the one at the origin, and (2) the intersection of the hyperbola $xy = 2a^2$ and the parabola $x^2 = ay$. For a full discussion of these solutions and the development of conic sections at this period see Heath's introduction to his edition of the *Conics of Apollonius*, Cambridge, 1896.

[3] For a description of the mechanical device of Plato see Sturm, page 49 and ff.

[4] Nichomedes also used this curve to trisect an angle, Pappus, page 57.

[5] A description of these curves and the solutions by means of their aid is given in part II of this paper.

[6] For a description of the other solutions of the Greeks see Sturm, pages 17–97.

[7] Stifel used the theory of irrationals given in Book X of the *Elements*. See Sturm, page 113.

[8] See Sturm, pages 125–7 and Ball, page 290.

[9] A proportionatrix is defined as follows: Describe about C as center the semicircle ADB on the diameter AB, then with CB as diameter draw the semicircle CEB on the same side of the diameter as ADB. Then draw any line BED cutting these semicircles in the points E and D. Then on AB take $BF = BE$. Then describe about F as center with a radius FB a circle cutting DB in G. Then the locus of G for different positions of BED is a proportionatrix.

[10] For an outline of these solutions and for references to the original sources, see Ball, page 290.

[11] The geometry appeared in 1637. In 1649 a Latin edition was put out by Van Schooten. A modernized French edition was published at Paris in 1886 under the title La Géométrie de René Descartes.

[12] Plane problems were those that could be solved by straight lines and circles, solid problems those that required conic sections, and linear problems included all others. See Pappus, pages 54 and 270.

Then in the third book Descartes takes up a discussion of the duplication and trisection problems and shows that the solution of all irreducible equations of the third degree leads to the solution of one or the other of these problems. His procedure is as follows: The intersection (x, y) of the parabola $x^2 = ay$ and the circle $x^2 + y^2 = ax + by$ is such that $a : x = x : y = y : b$. Thus the equation $x^3 = a^2b$ is given by the intersection of a parabola and a circle. But the parabola $y^2 = \frac{1}{4}x$ and the circle $x^2 + y^2 - \frac{13}{4}x + 4ay = 0$ intersect in a point which solves the trisection problem. The ordinate of this point of intersection is given by the equation $4y^3 = 3y - a$. Therefore a parabola and a circle will solve an equation ot the third degree, provided the squared term is missing. But since every cubic may be reduced to this form, every cubic may be solved by the intersection of a parabola and a circle, which is the same thing as solving either the duplication or the trisection problem.[1]

II. Solutions of Nicomedes and Diocles.

Solution of Nicomedes.[2] Nicomedes first considers the problem of constructing two mean proportionals between any two lines of given length. He proceeds as follows. Let CD and DA be two straight lines (Fig. 1). Complete the rectangle $ABCD$, bisect the lines AB and BC in the points L and E. Produce

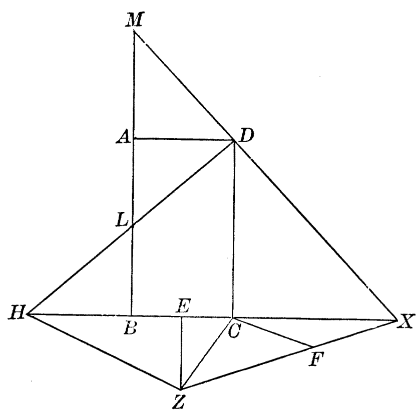


Fig. 1.

the lines DL and CB until they intersect in H. Draw EZ perpendicular to BC so that $ZC = AL$. Then draw ZH and CF parallel to ZH (F being indeterminate). Extend BC to X so that when a straight line is drawn from Z to X, $FX = AL = ZC$. This can be done by means of the conchoid.[3] Then draw

[1] See Sturm, pages 127–131 and Ball, pages 290 and 292.

[2] Pappus, Vol. I, pages 59–63 and pages 249–251.

[3] Z is the fixed point and CF is the fixed line of the conchoid, which is constructed as follows. Let there be a point Z and a line CF each given in position, and a line AL given in magnitude. Then let there be drawn through Z any line as ZX cutting CF in F and let $FX = AL$. The locus of X will be the curve called the conchoid of Nichomedes. There will of course be two branches to the curve, one on each side of CF.

XD and extend it until it meets AB produced in M. Then

$$DC : CX = CX : MA = MA : AD.$$

Proof: Since BC is bisected in E and CX is an extension of BC, then $BX \cdot XC + CE^2 = EX^2$ (Elements, II, 6).

Then by adding EZ^2, $BX \cdot XC + CE^2 + EZ^2 = EX^2 + EZ^2$, or $BX \cdot XC + CZ^2 = ZX^2$. Then by similar triangles, $MA : AB = MD : DX$ and $MD : DX = BC : CX$.

Therefore, $MA : AB = BC : CX$. But $AB = 2AL$ and $BC = \frac{1}{2}HC$, so that $MA : AL = HC : CX$.

Therefore, $MA : AL = ZF : FX$, and by composition $ML : AL = ZX : FX$. But by hypothesis, $AL = FX$, therefore $ML = ZX$, and $ML^2 = ZX^2$. Also (*Elements*, II, 6) $ML^2 = BM \cdot MA + AL^2$, and we have proved above that $ZX^2 = BX \cdot XC + CZ^2$ and $AL^2 = CZ^2$.

Therefore, $BM \cdot MA = BX \cdot XC$, or $BM : BX = CX : MA$.

But on account of parallels MB and DC, $BM : BX = DC : CX$.

Therefore, $DC : CX = CX : MA$.

Also on account of parallels BX and AD, $MB : BX = MA : AD$.

But from above $BM : BX = CX : MA$. Hence $CX : MA = MA : AD$.

Therefore, $DC : CX = CX : MA = MA : AD$. Thus CX and MA are the two required mean proportionals between DC and AD.

From this the duplication problem follows readily, as in note 2, page 107. Because if we have two given lines a and b and if c and d are the two mean proportionals to these lines, then by *Elements, V, Def. 11*; VIII, 12; XI, 33, $a : b = a^3 : c^3$. And if b is double a then c^3 is double a^3.

Solution of Diocles.[1] This solution is different from others because it does not find directly two mean proportionals to two given lines, but, after setting up a cissoid, finds two mean proportionals to two lines that have to each other a ratio equal to the ratio of the two given lines to each other, and from these auxiliary mean proportionals finds those required.

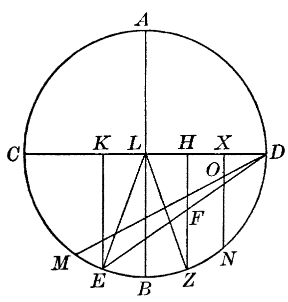
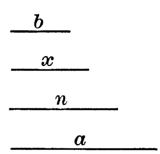


FIG. 2.



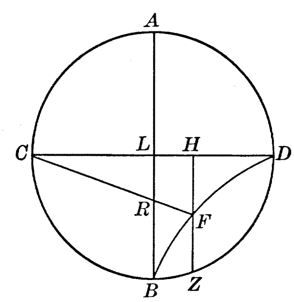


FIG. 3.

Let there be drawn in a circle two diameters perpendicular to each other as AB and CD (Fig. 2) and on either side of B cut off equal arcs EB and BZ, and draw ZH parallel to AB, and draw DE cutting ZH at F. Then ZH and HD will be two mean proportionals between CH and HF.

Proof: Let EK be drawn parallel to AB. Then $EK = ZH$ and $KC = HD$, as is seen if we draw EL and ZL.

For $\angle CLE = \angle ZLD$ and the angles at K and H are right angles. Consequently the triangles ELK and ZLH are congruent, since $LE = LZ$.

[1] This is on the authority of Eutocius, Archimedes, *Opera*, Vol. III, p. 78 ff.

Therefore, $KL = LH$ and hence $CK = HD$.

Since $DK : KE = DH : HF$, on account of similar triangles, and since $DK : KE = KE : KC$, for KE is the mean proportional between DK and KC, then $DK : KE = KE : KC = DH : HF$.

Then, since $DK = CH$, $KE = ZH$ and $KC = HD$, $CH : HZ = HZ : HD = HD : HF$.

This proves that ZH and HD are two mean proportionals between CH and HF.

Again if any two other equal arcs BM and BN are cut off and NX drawn parallel to AB, and DM is drawn cutting NX in O, then by reasoning similar to the above, NX and XD are the two mean proportionals between CX and XO. (The locus of all points determined, such as F and O, is called a cissoid.)[1]

Let there be now two given lines a and b, between which it is desired to find two mean proportionals (Fig. 3). Let there be constructed in a circle the two diameters CD and AB perpendicular to each other, and let the cissoid BFD be described. Let L be the center of the circle and R a point on the radius LB such that $a : b = CL : LR$. Draw CR and produce it until it cuts the cissoid in the point F, and draw through F the line HZ parallel to AB, cutting CD in H and the circle in Z. Then HZ and HD are two mean proportionals between CH and HF. (This has been proved above.)

Since $CH : HF = CL : LR$ and $CL : LR :: a : b$, it is only necessary to insert between a and b two lines, as n and x, so that these four will be in the same ratio as CH, HZ, HD and HF. Then n and x are the two mean proportionals between a and b.[2]

Then, as was shown above in the solution of Nicomedes, $a : b = a^3 : n^3$.

Pappus[3] also gives a solution which is essentially the same as that of Diocles given above. The only difference between the two solutions is this: Pappus, instead of using a curve to determine F, revolves a ruler about D until the portion between CR and RB equals the segment cut off by the line RB and the arc BC. But this is equivalent to laying off equal arcs on either side of AB.

III. Solution of Pappus.

The following is a discussion of an approximate solution proposed to Pappus (end of the third century A.D.) by an unknown geometer. It is recorded in Book III of the *Collections*[4] and is here reproduced with slight modifications of a literal translation. The construction without proof was handed over to Pappus, who shows in his discussion of it, that, although rather long, the construction does nothing more than assume that two mean proportionals are inserted between two given lines (a thing considered impossible by all preceding mathematicians provided only ruler and compasses are allowed in the construction) and that, considered geometrically, it is inexact. Pappus did not seem to realize the fact that the construction affords a method for obtaining an approximate solution of the problem in question. The construction and criticism proceed as follows:

Proposed Construction. Let there be two straight lines AB and AC perpendicular to each other (Fig. 4), and let there be drawn from B a line BD parallel to AC and let $BD = AB$. Join DC and produce DC until it meets AB produced in E, and from E draw EF parallel to BD.

Produce BD, and from D draw a line parallel to BE and intersecting EF in H; and in BD produced take $DN = NL = LX = XK = BD$, and through the points N, L, X, K draw NO, LM, XP, KF intersecting EF in the points O, M, P, F.

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[1] Eutocius however does not give any name to the curve. On the other hand Pappus and Proclus both mention the name cissoid but do not mention the name of the inventor of the curve. Nevertheless, the properties of the curve described by Eutocius seem to coincide with those given by Proclus in connection with the curve which he calls the cissoid. See Sturm, page 87.

[2] The determination of n and x may be made in the following manner. $CH : HZ = a : n$, and $HZ : HD = n : x$, and from this it is evident that $HD : HF = x : b$.

[3] Pappus, pages 64, 167 and 1070.

[4] Pappus, p. 30 and ff.

On KF take a point R so that $KR = AB$, and let KR be bisected in S, and on KF let T and Q be points such that $KF : FS = FS : FT = FT : FQ$.

Then on the line XP take $XX' = AB$, and join $X'K$ and $X'Q$ and draw from the point S a line SY parallel to $X'Q$ and cutting $X'K$ in Y, and from Y the line YW parallel to XK cutting ML in the point W.

Then let there be on ML the points A' and B' such that $LM : MW = MW : MA' = MA' : MB'$. And on ON take $NC' = AB$ and draw $C'L$, $C'B'$, and WD' parallel to $C'B'$ and cutting $C'L$ in D'.

Then from D' draw $D'E'$ parallel to LN and cutting DH in E'. Let Z' and F' be two points on DH such that $DH : HE' = HE' : HZ' = HZ' : HF'$.

Now draw $F'C$ and draw $Z'K'$ and $E'L'$ parallel to $F'C$ and cutting DC in the points K' and L', respectively, and from the points K' and L' draw the lines $K'M'$ and $L'N'$ parallel to AC and intersecting AB in M' and N', respectively.

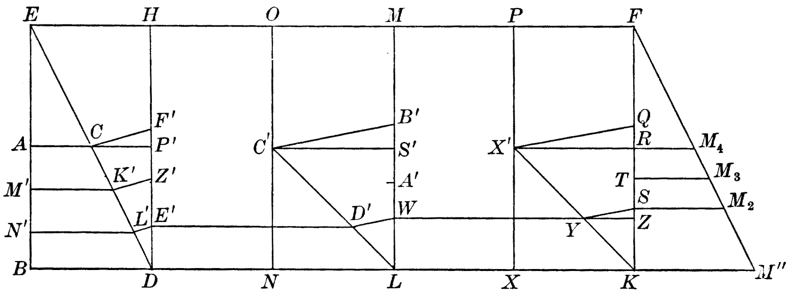

FIG. 4.

Then $M'K'$ and $N'L'$ are two mean proportionals between AC and BD.

Discussion of Pappus. Now the position of the point Q depends upon the ratio $KF : FR = BE : EA$ and hence Q may fall between F and R or between R and T, as we shall show.[1]

First let the ratio be double, that is, $KF : FR = 2 : 1 = 4 : 2$. Then $KF : FS = 4 : 3$ (for by construction $RS = \frac{1}{2}RK$), and from hypothesis $FS : FT = KF : FS$, that is, $KF : FS = FS : FT$ or $4 : 3 = 3 : 2\frac{1}{4}$. But $FS : FT = FT : FQ$, that is, $3 : 2\frac{1}{4} = 2\frac{1}{4}$: a number less than 2. If, however, we take $FR = 2$, then $FQ < FR$, and Q falls between the points F and R.

Again, let the ratio be quadruple, then $KF : FR = 8 : 2$, and so $KF : FS = 8 : 5$. For, if we suppose that KF contains 8 units and FR 2 units, then $RK = 6$ units and $RS = 3$ units, and, therefore, $FR + RS = FS = 5$ units. But $8 : 5 = 5 : 3\frac{1}{8}$, and $5 : 3\frac{1}{8} = 3\frac{1}{8}$: a number less than 2, so that, as above, Q will fall between F and R.

Finally, let the ratio be quintuple, that is, $KF : FR = 10 : 2$. Then $KF : FS = 10 : 6$. And $10 : 6 = 6 : 3\frac{3}{5}$, and $6 : 3\frac{3}{5} = 3\frac{3}{5}$: number greater than 2. Therefore Q falls between the points R and T. (For $FQ > FR$. But the same $FQ < FT$, since by hypothesis $KF : FS = FT : FQ$. Therefore Q falls between R and T.)

Now, since we have proved that Q may fall between F and R or between R and T, we will consider first that, no matter where Q may be assumed, $SF : FT = FT : FR$ is not possible, for that is assuming the thing sought. For let the line XK be produced to M'' so that $KM'' = XK$ and let $M''F$ be drawn and lines parallel to KM'' through the points S, T, and R intersecting FM'' in the points M_2, M_3, M_4, and the problem will be completed as is manifest. Because $KM'' : SM_2 = SM_2 : TM_3 = TM_3 : RM_4$, and $KM'' = BD$ and $RM_4 = AC$. Therefore, SM_2, and TM_3 are two mean proportionals between BD and AC, a thing which is impossible. For since FK is a straight line and in it is a point R, it is not possible by means of the ratios of plane figures to assume two points T and S in FK such that $KF : FS = FS : FT = FT : FR$; for the problem is by nature solid.[2]

Therefore, the proposer did not dare to say that the point Q would fall on R, for that would assume the whole problem, but he has chosen Q between F and R and then finished the construc-

[1] This is shown by special numbers. The method is of interest because it shows how the Greeks were forced to attack problems of this nature because of a lack of algebra.

[2] See note 12, p. 108. Pappus here as elsewhere makes the statement that this problem, as well as the trisection problem, is not possible by means of straight lines and circles. It is in no wise probable that the Greeks ever proved this statement. See Pappus, p. 271.

tion arbitrarily, and gradually fallen back into the first difficulty. He has produced a long construction, not in order to deceive his readers, but he has led himself into error, as I shall show after I have considered the problem in a sensible manner.

Now since the ratio $KF : FR$ is given and FK is given (for it is necessary that each be assumed), then FR is given (*cf.* Euclid's work, *The Data*, proposition 2), therefore the remainder KR is given (*Data*, 4). But SR is given because it is half of RK, and also FR is given, therefore FS is given (*Data*, 3). Therefore the ratio $KF : FS$ is given (*Data*, 1). And by hypothesis $KF : FS = FS : FT$, and we have proved that FS is given, therefore FT will be given. And in the same way FQ will be given. Therefore the difference $FR - FQ$ will be given.

Now suppose the point Q to be between F and R. We have proved this to be possible. Then since the difference $FR - FQ = QR$ is given, and the line joining X' and R is given, because it equals XK, the right triangle $QX'R$ is given in species and magnitude.[1] Therefore the angle RQX' is given, and because of the parallels $X'Q$ and YS we have $\angle RQX' = \angle KSY$.

Now let the line WY be produced till it intersects FK in Z. Then the triangle SZY is given in species (*Data*, 40). But it is also given in magnitude.[2] Therefore YZ is given. Therefore YZ, which is parallel to XK, is in the same straight line with WY. Therefore $WL = ZK$ is given (opposite sides of a parallelogram). Then, because $KF = LM$, and $SK > WL$, for $WL = ZK$, $FS < MW$. Also $KF : FS = FS : FT = FT : FQ$ and $LM : MW = MW : MA' = MA' : MB'$. Therefore $FQ < MB'$.[3] Hence, $KF - FQ > LM - MB'$, that is $B'L < QK$.

Then, again, because WL is given (proved above) and LM is given (since it equals KF which was given), MW is given, and from this the ratio $LM : MW$ is given. And by hypothesis $LM : MW = MW : MA'$, and MW is given, therefore MA' is given.

By similar reasoning MB' may be proved to be given, and from this the point B' is given, and may be between the points S' and M or between S' and A' (S' being a point on ML such that $S'L = KR = AB$).

But if the point B' is assumed to fall on S' it is equivalent to assuming the problem. For, again, in the line ML in which there is a given point S', two points A' and B' are assumed such that $LM : MW = MW : MA' = MA' : MS'$, a thing that no one will grant to be possible. Then if $C'S'$ is drawn, as above, the triangle $S'B'C'$ will be given in species and magnitude and finally DE' will be given. And DH is given and DE' is given, therefore HE' is given, and the ratio $DH : HE'$ is given, and $DH : HE' = HE' : HZ' = HZ' : HF'$. Then (if $DP' = KR$) F' may fall between P' and Z' or P' and H, but not on P' for that is equivalent to assuming the whole problem. But wherever F' may be, let it be supposed to be above P'.

Then when $F'C$ is drawn and parallel to $F'C$ the two lines $Z'K'$ and $E'L'$, and through the points K' and L' parallel to AC the two lines $K'M'$ and $L'N'$ are drawn, it is clear that the problem is not solved. For since $F'C$ is not parallel to EH the angle $CF'H$ will be obtuse if F' falls between P' and H, and acute if F' falls between P' and Z'. But if CP' is drawn, the angle at P' is a right angle. Therefore, the problem will be solved only if F' falls on P', so that in the line DH there are two points E' and Z' such that $DH : HE' = HE' : HZ' = HZ' : HP'$. But if this is not granted the problem can not be solved by means of planes.[4] And this will be persuasive to all by means of numbers, if they admit the table of Ptolemy of lines in a circle.[5] In fact it would have been more satisfactory if he, like the rest, had left the solution in doubt instead of producing it in this manner.

Such is the comment of Pappus. It remained for Günther and Pendlebury, in comparatively recent times, to exhibit the construction of the unknown geometer as one of a series of approximations leading to an accurate solution.[6]

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[1] A figure is said to be given in species if its angles and the ratios of its sides are given.

[2] For YS is given because $X'Q : YS = KQ : KS$. From which it readily follows that the triangle YZS is given in magnitude.

[3] For proof of this statement see Pappus, p. 51.

[4] By a "solution by planes" is meant a solution by means of straight lines and circles.

[5] This is the table of chords of Ptolemy given in the *Almagest*. This table uses the measure of the chords to represent the measure of the arcs subtended by the chords. The lengths of the chords of certain arcs in a given circle were known, and, using these to start with, the lengths of the chords of the sum or difference of two arcs or of the half arcs were found. For the method of doing this, see Gow, *Short History of Greek Mathematics*, page 294 f.

[6] For a further discussion of this construction see Sturm, page 94 ff.